

On the Distribution of Cube-Free Numbers with the Form $[n^c]$

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Abstract: In this paper, we proved that there are infinite cube-free numbers of the form $[n^c]$ for any fixed real number $1 < c < 11/6$.

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1 Introduction and main result

Let \mathbb{N} denote the set of natural numbers, \mathcal{P} denote the set of all primes and \mathfrak{F}_k denote the set of k -free numbers. For any real number $c > 0$ and any infinite subset \mathcal{A} of \mathbb{N} , define

$$\mathcal{A}^c := \{n \in \mathbb{N} : n = [m^c], m \in \mathcal{A}\}.$$

Let \mathcal{A}, \mathcal{B} be two infinite subset of \mathbb{N} and c be a fixed real number. It is an important problem to investigate that whether $\mathcal{A}^c \cap \mathcal{B}$ is a infinite subset of \mathbb{N} or not. For this problem, several cases have been studied.

Case 1. $\mathcal{A} = \mathbb{N}, \mathcal{B} = \mathcal{P}$.

Piatetski-Shapiro [9] first studied this case. He proved that the set $\mathbb{N}^c \cap \mathcal{P}$ is infinite for $0 < c < 12/11$. This result is called *Piatetski-Shapiro Prime Number Theory*. If $0 < c \leq 1$, this result is the simple corollary of PNT. However, for $1 < c < 12/11$, the conclusion is not trivial. Later, the exponent $12/11$ was improved by many authors, for historical literatures the reader should consult Rivat and Wu [10] and its references.

Case 2. $\mathcal{A} = \mathcal{B} = \mathcal{P}$.

This case is quite difficult, even though $0 < c < 1$ is not trivial. Balog [1] proved that the set $\mathcal{P}^c \cap \mathcal{P}$ is infinite for $0 < c < 5/6$. Although Balog proved that, for almost

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all $c > 1$, the set $\mathcal{P}^c \cap \mathcal{P}$ is infinite in the sense of Lebesgue measure, he can not prove that $\mathcal{P}^c \cap \mathcal{P}$ is infinite for each fixed $c > 1$.

Case 3. $\mathcal{A} = \mathbb{N}$, $\mathcal{B} = \mathfrak{F}_2$.

Rieger [11] first investigated this case. He proved that the set $\mathbb{N}^c \cap \mathfrak{F}_2$ is infinite for $1 < c < 3/2$, which can be derived from the result of Deshouillers [6]. More precisely, Rieger proved that the asymptotic formula

$$\mathbb{N}^c \cap \mathfrak{F}_2(x) := \sum_{\substack{n \leq x \\ [n^c] \in \mathfrak{F}_2}} 1 = \frac{6}{\pi^2} x + O(x^{\frac{2c+1}{4} + \varepsilon})$$

holds for $1 < c < 3/2$. In 1998, Cao and Zhai proved that the asymptotic formula

$$\mathbb{N}^c \cap \mathfrak{F}_2(x) = \frac{6}{\pi^2} x + O(x^{\frac{36(c+1)}{97} + \varepsilon})$$

holds for $1 < c < 61/36$. It is important to emphasise that Stux [13] proved that the set $\mathbb{N}^c \cap \mathfrak{F}_2$ is infinite for almost all $c \in (1, 2)$ in the sense of Lebesgue measure. However, his method can not determine the value of c such that $\mathbb{N}^c \cap \mathfrak{F}_2$ is infinite. In 2008, Cao and Zhai [5] improved their earlier result in [3] and show that, for any fixed $1 < c < 149/87$, the set $\mathbb{N}^c \cap \mathfrak{F}_2$ is infinite.

In this paper, we consider the case $\mathcal{A} = \mathbb{N}$, $\mathcal{B} = \mathfrak{F}_3$. To be specific, we shall prove that, for a class of infinite sets $\mathcal{A} \subseteq \mathbb{N}$, $\mathcal{A}^c \cap \mathfrak{F}_3$ are infinite sets.

Let $c > 1$ be a real number and $\mathcal{A} \subseteq \mathbb{N}$ satisfying the following two conditions:

(1) For any $\eta > 0$, there holds

$$\mathcal{A}(x) := \sum_{\substack{n \leq x \\ n \in \mathcal{A}}} 1 \gg x^{1-\eta}; \quad (1.1)$$

(2) Let $\eta > 0$ be an arbitrary small real number and $x > 1$ be a real number. If $\alpha = a/q$ is a rational number satisfying $(a, q) = 1$ and $2 \leq q \leq x^\eta$, then there exists positive constant δ , which depends only on c , satisfying $\eta \leq \delta < 1/2$ such that there holds

$$\sum_{\substack{n \leq x \\ n \in \mathcal{A}}} e(\alpha[n^c]) \ll x^{1-\delta}. \quad (1.2)$$

Remark There are many subsets of \mathbb{N} satisfying (1.1) and (1.2). For instance, the sets \mathbb{N} , \mathcal{P} , \mathfrak{F}_k ($k = 2, 3, \dots$), etc.

The main result is the following theorem.

Theorem 1.1 *Let $1 < c < 11/6$, $\gamma = c^{-1}$ and $0 < \varepsilon < 10^{-10}$ be a sufficiently small constant. Suppose that the set $\mathcal{A} \subseteq \mathbb{N}$ satisfies the condition (1.1) and (1.2). Then we have*

$$\mathcal{A}^c \cap \mathfrak{F}_3(x) := \sum_{\substack{n \leq x \\ n \in \mathcal{A}, [n^c] \in \mathfrak{F}_3}} 1 = \frac{1}{\zeta(3)} \mathcal{A}(x) + O(x^{1-\varepsilon}).$$

Corollary 1.2 *Let $1 < c < 11/6$, $\gamma = c^{-1}$ and $0 < \varepsilon < 10^{-10}$ be a sufficiently small constant. Then we have*

$$\begin{aligned}\mathbb{N}^c \cap \mathfrak{F}_3(x) &= \frac{x}{\zeta(3)} + O(x^{1-\varepsilon}), \\ \mathcal{P}^c \cap \mathfrak{F}_3(x) &= \frac{1}{\zeta(3)} \int_2^x \frac{du}{\log u} + O(xe^{-c_1\sqrt{\log x}}), \\ \mathfrak{F}_3^c \cap \mathfrak{F}_3(x) &= \frac{x}{\zeta^2(3)} + O(x^{1-\varepsilon}),\end{aligned}$$

where c_1 is an absolute constant.

Notation In this paper, we use $[x]$, x and $\|x\|$ to denote the integral part of x , the fractional part of x and the distance from x to the nearest integer, respectively; $\mu(n)$ denotes Möbius function; $e(t) = e^{2\pi it}$; $\psi(x) = x - [x] - 1/2$; $n \sim N$ denotes $N < n \leq 2N$.

2 Preliminary Lemmas

In order to prove Theorem we need the following two lemmas.

Lemma 2.1 *For any $J \geq 2$, we have*

$$\psi(t) = \sum_{1 \leq |h| \leq J} a(h)e(ht) + O\left(\sum_{|h| \leq J} b(h)e(ht)\right), \quad a(h) \ll \frac{1}{|h|}, \quad b(h) \ll \frac{1}{J}.$$

Proof. See pp. 116 of Graham and Kolesnik [7] or Vaaler [14]. ■

Lemma 2.2 *For any $H \geq 1$, we have*

$$\psi(\theta) = - \sum_{0 < |h| \leq H} \frac{e(\theta h)}{2\pi i h} + (g(\theta, H)),$$

where

$$g(\theta, H) = \min\left(1, \frac{1}{H\|\theta\|}\right) = \sum_{h=-\infty}^{+\infty} a_h e(\theta h)$$

and

$$a_h \ll \min\left(\frac{\log 2H}{H}, \frac{1}{|h|}, \frac{H}{|h|^2}\right).$$

Proof. See pp. 245 of Heath-Brown [8]. ■

Lemma 2.3 *Let y be not an integer, $\alpha \in (0, 1)$, $H \geq 3$. Then we have*

$$e(-\alpha\{y\}) = \sum_{|h| \leq H} c_h(\alpha)e(hy) + O\left(\min\left(1, \frac{1}{H\|y\|}\right)\right),$$

where

$$c_h(\alpha) := \frac{1 - e(-\alpha)}{2\pi i(h + \alpha)}.$$

Proof. See the thesis of Buriev [2]. ■

3 Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. Let c and ε satisfy the conditions in Theorem 1.1. It is well known that the characteristic function of cube-free numbers is

$$\sum_{d^3|n} \mu(d) = \begin{cases} 0, & \exists m \text{ s.t. } m^3|n, \\ 1, & \text{others.} \end{cases}$$

Then, we can write

$$\begin{aligned} \mathcal{A}^c \cap \mathfrak{F}_3(x) &:= \sum_{\substack{n \leq x \\ n \in \mathcal{A}, [n^c] \in \mathfrak{F}_3}} 1 = \sum_{\substack{n \leq x \\ n \in \mathcal{A}}} \sum_{d^3|[n^c]} \mu(d) \\ &= \sum_{\substack{n \leq x \\ n \in \mathcal{A}}} \sum_{\substack{d^3|[n^c] \\ d \leq x^\varepsilon}} \mu(d) + \sum_{\substack{n \leq x \\ n \in \mathcal{A}}} \sum_{\substack{d^3|[n^c] \\ d > x^\varepsilon}} \mu(d) \\ &=: \Sigma_1 + \Sigma_2. \end{aligned} \tag{3.1}$$

From the formula

$$\sum_{h=1}^q e\left(\frac{hn}{q}\right) = \begin{cases} q, & \text{if } q \mid n, \\ 0, & \text{if } q \nmid n, \end{cases}$$

we get

$$\begin{aligned} \Sigma_1 &= \sum_{\substack{m \leq x \\ m \in \mathcal{A}}} \sum_{d \leq x^\varepsilon} \frac{\mu(d)}{d^3} \sum_{\ell=1}^{d^3} e\left(\frac{\ell[m^c]}{d^3}\right) = \sum_{d \leq x^\varepsilon} \frac{\mu(d)}{d^3} \sum_{\ell=1}^{d^3} \sum_{\substack{m \leq x \\ m \in \mathcal{A}}} e\left(\frac{\ell[m^c]}{d^3}\right) \\ &= \sum_{d \leq x^\varepsilon} \frac{\mu(d)}{d^3} \mathcal{A}(x) + \sum_{2 \leq d \leq x^\varepsilon} \frac{\mu(d)}{d^3} \sum_{\ell=1}^{d^3-1} \sum_{\substack{m \leq x \\ m \in \mathcal{A}}} e\left(\frac{\ell[m^c]}{d^3}\right). \end{aligned} \tag{3.2}$$

Taking $\eta = 2\varepsilon$ in (1.2), then there exists δ satisfying $2\varepsilon \leq \delta \leq 1/2$. From (1.2) we obtain

$$\sum_{2 \leq d \leq x^\varepsilon} \frac{\mu(d)}{d^3} \sum_{\ell=1}^{d^3-1} \sum_{\substack{m \leq x \\ m \in \mathcal{A}}} e\left(\frac{\ell[m^c]}{d^3}\right) \ll x^{1-\delta} \sum_{2 \leq d \leq x^\varepsilon} \frac{d^3-1}{d^3} \ll x^{1-\varepsilon}.$$

It is easy to see that

$$\sum_{d \leq x^\varepsilon} \frac{\mu(d)}{d^3} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^3} + O(x^{-2\varepsilon}) = \frac{1}{\zeta(3)} + O(x^{-2\varepsilon}). \tag{3.3}$$

From (3.3) and the fact that $\mathcal{A} \subseteq \mathbb{N}$, we get

$$\Sigma_1 = \frac{1}{\zeta(3)} \mathcal{A}(x) + O(x^{1-2\varepsilon}). \tag{3.4}$$

Now we estimate Σ_2 . We have

$$\begin{aligned}
\Sigma_2 &= \sum_{\substack{m \leq x \\ m \in \mathcal{A}}} \sum_{\substack{d^3 \ell \leq m^c < d^3 \ell + 1 \\ d > x^\varepsilon}} \mu(d) \ll \sum_{m \leq x} \sum_{\substack{d^3 \ell - 2 < m^c \leq d^3 \ell + 2 \\ d > x^\varepsilon}} 1 \\
&\ll \sum_{m \leq x} \sum_{\substack{(d^3 \ell - 2)^\gamma < m \leq (d^3 \ell + 2)^\gamma \\ d > x^\varepsilon}} 1 \ll \sum_{\substack{d^3 \ell \leq x^c \\ d > x^\varepsilon}} \left([(d^3 \ell + 2)^\gamma] - [(d^3 \ell - 2)^\gamma] \right) \\
&\ll (\log x)^2 \sum_{d \sim D} \sum_{\ell \sim L} \left([(d^3 \ell + 2)^\gamma] - [(d^3 \ell - 2)^\gamma] \right) \tag{3.5}
\end{aligned}$$

for some pair (D, L) , where $x^\varepsilon \ll D \ll x^{c/3}$, $1 \ll L \ll x^{c-3\varepsilon}$, $D^3 L \ll x^c$.

If $d \sim D$, $\ell \sim L$, then, by Lagrange's mean value theorem, we get

$$(d^3 \ell + 2)^\gamma - (d^3 \ell)^\gamma < 2\gamma(d^3 \ell)^{\gamma-1} < 2\gamma(D^3 L)^{\gamma-1} < 4\gamma(D^3 L)^{\gamma-1}$$

and

$$(d^3 \ell)^\gamma - (d^3 \ell - 2)^\gamma < 2\gamma(d^3 \ell - 2)^{\gamma-1} < 2\gamma(d^3 \ell/2)^{\gamma-1} < 4\gamma(D^3 L)^{\gamma-1}.$$

Therefore, we obtain

$$\begin{aligned}
&\sum_{d \sim D} \sum_{\ell \sim L} \left([(d^3 \ell + 2)^\gamma] - [(d^3 \ell - 2)^\gamma] \right) \\
&= \sum_{\substack{d \sim D, \ell \sim L \\ (d^3 \ell - 2)^\gamma < m \leq (d^3 \ell + 2)^\gamma}} 1 \ll \sum_{\substack{d \sim D, \ell \sim L \\ (d^3 \ell)^\gamma - 4\gamma(D^3 L)^{\gamma-1} < m \leq (d^3 \ell)^\gamma + 4\gamma(D^3 L)^{\gamma-1}}} 1 \\
&= \sum_{d \sim D} \sum_{\ell \sim L} \left([(d^3 \ell)^\gamma + 4\gamma(D^3 L)^{\gamma-1}] - [(d^3 \ell)^\gamma - 4\gamma(D^3 L)^{\gamma-1}] \right) \\
&\ll (D^3 L)^{\gamma-1} DL + \left| \sum_{d \sim D} \sum_{\ell \sim L} \psi((d^3 \ell)^\gamma - 4\gamma(D^3 L)^{\gamma-1}) \right| \\
&\quad + \left| \sum_{d \sim D} \sum_{\ell \sim L} \psi((d^3 \ell)^\gamma + 4\gamma(D^3 L)^{\gamma-1}) \right| \\
&=: (D^3 L)^{\gamma-1} DL + |\mathcal{T}_+(D, L)| + |\mathcal{T}_-(D, L)| \\
&\ll x^{1-2\varepsilon} + |\mathcal{T}_+(D, L)| + |\mathcal{T}_-(D, L)|, \tag{3.6}
\end{aligned}$$

where the last step uses the following estimate

$$(D^3 L)^{\gamma-1} DL = (D^3 L)^{\gamma-1} D^3 L \cdot D^{-2} = (D^3 L)^\gamma \cdot D^{-2} \ll x^{1-2\varepsilon}.$$

From (3.1), (3.4), (3.5) and (3.6), we can see that it is sufficient to show

$$\mathcal{T}_+(D, L) \ll x^{1-2\varepsilon}, \quad \mathcal{T}_-(D, L) \ll x^{1-2\varepsilon}. \tag{3.7}$$

Set $N := D^3 L$. We shall prove

$$\mathcal{T}_+(D, L) \ll N^{\gamma-2\varepsilon}, \quad \mathcal{T}_-(D, L) \ll N^{\gamma-2\varepsilon}, \tag{3.8}$$

from which we can deduce (3.7) immediately. If $D \gg N^{(1-\gamma+2\varepsilon)/2}$, then we have

$$\mathcal{T}_{\pm}(D, L) \ll DL = D^3 L \cdot D^{-2} = ND^{-2} \ll N^{1-(1-\gamma+2\varepsilon)} = N^{\gamma-2\varepsilon}.$$

Thus, from what follows, we always assume that $D \ll N^{(1-\gamma+2\varepsilon)/2}$ and $DL \geq 100N^{\gamma-2\varepsilon}$.

Taking $J = \lfloor D^2 L N^{2\varepsilon-\gamma} \rfloor$ in Lemma 2.1, then we have

$$\begin{aligned} \mathcal{T}_{\pm}(D, L) &= \sum_{d \sim D} \sum_{\ell \sim L} \left(\sum_{1 \leq |h| \leq J} a(h) e\left(h((d^3 \ell)^{\gamma} \pm 4\gamma(D^3 L)^{\gamma-1})\right) \right. \\ &\quad \left. + O\left(\sum_{|h| \leq J} b(h) e\left(h((d^3 \ell)^{\gamma} \pm 4\gamma(D^3 L)^{\gamma-1})\right)\right) \right) \\ &= \mathbf{I} + \mathbf{II}, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \mathbf{I} &= \sum_{1 \leq |h| \leq J} a(h) \sum_{d \sim D} \sum_{\ell \sim L} e\left(h((d^3 \ell)^{\gamma} \pm 4\gamma(D^3 L)^{\gamma-1})\right) \\ &\ll \sum_{1 \leq h \leq J} \frac{1}{h} \left| \sum_{d \sim D} \sum_{\ell \sim L} e(h(d^3 \ell)^{\gamma}) \right|, \end{aligned} \tag{3.10}$$

$$\begin{aligned} \mathbf{II} &\ll \sum_{d \sim D} \sum_{\ell \sim L} \sum_{|h| \leq J} b(h) e\left(h((d^3 \ell)^{\gamma} \pm 4\gamma(D^3 L)^{\gamma-1})\right) \\ &\ll \frac{DL}{J} + \sum_{1 \leq h \leq J} |b(h)| \left| \sum_{d \sim D} \sum_{\ell \sim L} e(h(d^3 \ell)^{\gamma}) \right| \\ &\ll N^{\gamma-2\varepsilon} + \sum_{1 \leq h \leq J} \frac{1}{J} \left| \sum_{d \sim D} \sum_{\ell \sim L} e(h(d^3 \ell)^{\gamma}) \right| \\ &\ll N^{\gamma-2\varepsilon} + \sum_{1 \leq h \leq J} \frac{1}{h} \left| \sum_{d \sim D} \sum_{\ell \sim L} e(h(d^3 \ell)^{\gamma}) \right|. \end{aligned} \tag{3.11}$$

Combining (3.9), (3.10) and (3.11), we derive

$$\begin{aligned} \mathcal{T}_{\pm}(D, L) &\ll N^{\gamma-2\varepsilon} + \sum_{1 \leq h \leq J} \frac{1}{h} \left| \sum_{d \sim D} \sum_{\ell \sim L} e(h(d^3 \ell)^{\gamma}) \right| \\ &\ll N^{\gamma-2\varepsilon} + \frac{1}{H} |\mathcal{S}(H, D, L)| \cdot \log J, \end{aligned} \tag{3.12}$$

for some $1 \ll H \ll J$, where

$$\mathcal{S}(H, D, L) := \sum_{h \sim H} \left| \sum_{d \sim D} \sum_{\ell \sim L} e(h(d^3 \ell)^{\gamma}) \right|.$$

Therefore, we only need to show

$$\mathcal{S}(H, D, L) \ll HN^{\gamma-2\varepsilon}. \quad (3.13)$$

Let $F = HN^\gamma$. Thus, we have $N^\gamma \ll F \ll D^2LN^{2\varepsilon}$. Next, in order to prove (3.13), we shall consider the following three cases.

Case 1 If $D \ll N^{2\gamma-1-6\varepsilon}$, we use exponential pair $(1/2, 1/2)$ to estimate the inner sum over ℓ and apply trivial estimate to the sum over h and d . Thus, we get

$$\begin{aligned} \mathcal{S}(H, D, L) &\ll \frac{HD}{HD^{3\gamma}L^{\gamma-1}} + HD((HD^{3\gamma}L^{\gamma-1})^{1/2}L^{1/2}) \\ &= \frac{DL}{N^\gamma} + HD((HN^\gamma L^{-1})^{1/2}L^{1/2}) \\ &\ll N^{1-\gamma} + HD(JN^\gamma)^{1/2} \ll N^{1-\gamma} + HD(D^2L)^{1/2}N^\varepsilon \\ &= N^{1-\gamma} + HD^{1/2}(D^3L)^{1/2}N^\varepsilon \\ &\ll N^{1-\gamma} + HD^{1/2}N^{1/2+\varepsilon} \ll HN^{\gamma-2\varepsilon}. \end{aligned}$$

Case 2 If $N^{2\gamma-1-6\varepsilon} \ll D \ll N^{6\gamma-3-22\varepsilon}$, by Theorem 7 of Cao and Zhai [4] with parameters $(M, M_1, M_2) = (L, H, D)$, we obtain

$$\begin{aligned} N^{-\varepsilon} \cdot \mathcal{S}(H, D, L) &\ll (F^2L^3H^7D^7)^{1/8} + (F^4H^7D^7)^{1/8} + (F^{18}L^{15}H^{54}D^{54})^{1/58} \\ &\quad + (F^{35}L^{26}H^{100}D^{100})^{1/108} + (F^{31}L^{24}H^{92}D^{92})^{1/98} \\ &\quad + (F^{10}L^6H^{27}D^{27})^{1/29} + (F^{111}L^{86}H^{294}D^{294})^{1/336} \\ &\quad + (F^{103}L^{74}H^{266}D^{266})^{1/304} + (F^{119}L^{74}H^{294}D^{294})^{1/336} \\ &\quad + (F^{80}L^{19}H^{188}D^{188})^{1/200} + (F^{149}L^{34}H^{344}D^{344})^{1/368} \\ &\quad + (F^{43}L^5H^{94}D^{94})^{1/100} + (F^2LH^6D^6)^{1/6} \\ &\quad + (F^4L^{-1}H^8D^8)^{1/8} + F^{-1/2}LHD \\ &\ll HN^{\gamma-3\varepsilon}. \end{aligned}$$

Case 3 If $D \gg N^{6\gamma-3-22\varepsilon}$, by Theorem 3 of Robert and Sargos [12] with parameters $(H, N, M) = (H, D, L)$, we deduce that

$$\begin{aligned} N^{-2\varepsilon} \cdot \mathcal{S}(H, D, L) &\ll (HDL) \left(\left(\frac{F}{HDL^2} \right)^{1/4} + \frac{1}{L^{1/2}} + \frac{1}{F} \right) \\ &\ll HN^{\gamma/4}D^{3/4}L^{1/2} + HDL^{1/2} + DLN^{-\gamma} \ll HN^{\gamma-4\varepsilon} \end{aligned}$$

under the condition $D \gg N^{(2-3\gamma+16\varepsilon)/3}$. Moreover, by noting the fact that $\gamma > 6/11$, there must hold $N^{(2-3\gamma+16\varepsilon)/3} \ll N^{6\gamma-3-22\varepsilon}$.

Combining the above three cases, we have finished the proof of Theorem 1.1.

4 Proof of Corollary 1.2

In this section, we shall prove Corollary 1.2. Take $\eta = 2\varepsilon$ in (1.2). Suppose that $2 \leq d \leq x^\varepsilon$, $1 \leq \ell \leq d^2 - 1$, then there exist a pair of integers a and q satisfying $2 \leq q \leq x^\eta$, $1 \leq a \leq q - 1$, $(a, q) = 1$. Denote $\alpha = a/q$ and

$$S_c(x; \mathcal{A}, \alpha) := \sum_{\substack{n \leq x \\ n \in \mathcal{A}}} e(\alpha[n^c]).$$

In order to prove Corollary 1.2, we need to prove the following lemma.

Lemma 4.1 *Let $1 < c < 2$ and $0 < \eta(c - 1)/100$ be a sufficiently small constant, then we have*

$$S_c(x; \mathbb{N}, \alpha) \ll x^{1-(3-c)/7} \log x, \quad (4.1)$$

$$S_c(x; \mathcal{P}, \alpha) \ll x^{1-(5-2c)/90} \log^{19} x \quad (4.2)$$

$$S_c(x; \mathfrak{F}_3, \alpha) \ll x^{1-(11-4c)/22} \log^2 x. \quad (4.3)$$

Proof. We only need to prove (4.3), since (4.1) and (4.2) are from Lemma 2 of [5]. Obviously, it is sufficient to prove that the following estimate

$$S_c^*(N; \mathfrak{F}_3, \alpha) := \sum_{\substack{n \sim N \\ n \in \mathfrak{F}_3}} e(\alpha[n^c]) \ll N^{1-\delta} \log^\varpi N \quad (4.4)$$

holds with $x^{3/4} \ll N \ll x$.

Taking $H = N^\delta$, $\delta = (11 - 4c)/22$ in Lemma 2.3, we get

$$\begin{aligned} S_c^*(N; \mathfrak{F}_3, \alpha) &= \sum_{\substack{n \sim N \\ n \in \mathfrak{F}_3}} e(\alpha n^c - \alpha \{n^c\}) \\ &= \sum_{\substack{n \sim N \\ n \in \mathfrak{F}_3}} e(\alpha n^c) \left(\sum_{|h| \leq H} c_h(\alpha) e(hn^c) + O\left(\min\left(1, \frac{1}{H\|n^c\|}\right)\right) \right) \\ &= \sum_{|h| \leq H} c_h(\alpha) \sum_{\substack{n \sim N \\ n \in \mathfrak{F}_3}} e((h + \alpha)n^c) + O\left(\sum_{n \sim N} \min\left(1, \frac{1}{H\|n^c\|}\right)\right). \end{aligned}$$

From Lemma 2.2, we obtain

$$\begin{aligned} \sum_{n \sim N} \min\left(1, \frac{1}{H\|n^c\|}\right) &= \sum_{n \sim N} \sum_{h=-\infty}^{+\infty} a(h) e(hn^c) = \sum_{h=-\infty}^{+\infty} a(h) \sum_{n \sim N} e(hn^c) \\ &= N \cdot a(0) + \sum_{\substack{h=-\infty \\ h \neq 0}}^{+\infty} a(h) \sum_{n \sim N} e(hn^c) \\ &\ll N|a(0)| + \sum_{h=1}^{\infty} |a(h)| \left| \sum_{n \sim N} e(hn^c) \right| \end{aligned}$$

$$\begin{aligned}
&\ll N^{1-\delta} \log N + \sum_{h=1}^{\infty} \min\left(\frac{1}{h}, \frac{H}{h^2}\right) (hN^{c-1})^{4/18} N^{11/18} \\
&\ll N^{1-\delta} \log N + \sum_{h \leq H} h^{-7/9} N^{(4c+7)/18} + \sum_{h > H} \frac{H}{h^{16/9}} N^{(4c+7)/18} \\
&\ll N^{1-\delta} \log N + N^{(4c+7)/18} H^{2/9} \\
&\ll N^{1-\delta} \log N + N^{(4c+7+4\delta)/18} \ll N^{1-\delta} \log N,
\end{aligned}$$

where we use the exponential pair $(4/18, 11/18)$ to estimate the sum over n .

Since $2 \leq q \leq x^\eta$, $1 \leq a \leq q-1$, $(a, q) = 1$, then $x^{-\eta} \leq \alpha = a/q \leq (q-1)/q = 1 - 1/q \leq 1 - x^{-\eta}$. Thus, for $|h| \leq H$, there holds $x^{-\eta} \leq |h + \alpha| \ll N^\delta$. Noting that $0 < \eta < (c-1)/100$, we get $|h + \alpha|N^{c-1} \gg x^{-\eta}x^{3(c-1)/4} \gg 1$.

Therefore, we deduce that

$$\begin{aligned}
\sum_{\substack{n \sim N \\ n \in \mathfrak{F}_3}} e((h + \alpha)n^c) &= \sum_{n \sim N} \left(\sum_{m^3 | n} \mu(m) \right) e((h + \alpha)n^c) \\
&= \sum_{N < m^3 t \leq 2N} \mu(m) e((h + \alpha)m^{3c}t^c) \\
&= \sum_{\substack{N < m^3 n \leq 2N \\ m \leq N^\delta}} \mu(m) e((h + \alpha)m^{3c}n^c) + O(N^{1-2\delta}) \\
&= \sum_{m \leq N^\delta} \mu(m) \sum_{\substack{N/m^3 < n \leq 2N/m^3}} e((h + \alpha)m^{3c}n^c) + O(N^{1-2\delta}) \\
&\ll \sum_{m \leq N^\delta} \left(|h + \alpha| m^{3c} \left(\frac{N}{m^3} \right)^{c-1} \right)^{4/18} \left(\frac{N}{m^3} \right)^{11/18} + N^{1-2\delta} \\
&\ll N^{(4c+7+4\delta)/18} \left(\sum_{m \leq N^\delta} m^{-7/6} \right) + N^{1-2\delta} \\
&\ll N^{(4c+7+\delta)/18} + N^{1-2\delta} \ll N^{1-\delta},
\end{aligned}$$

where we use the exponential pair $(4/18, 11/18)$ to estimate the sum over n . From (26) of Cao and Zhai [5], we have

$$\sum_{|h| \leq H} c_h(\alpha) \sum_{\substack{n \sim N \\ n \in \mathfrak{F}_3}} e((h + \alpha)n^c) \ll N^{1-\delta} \log N.$$

This completes the proof of Lemma 4.1. ■

From the three following formulas

$$\begin{aligned}
\mathbb{N}(x) &= x + O(1), \\
\mathfrak{F}_3(x) &= \frac{x}{\zeta(3)} + O(x^{1/2+\varepsilon}), \\
\mathscr{P}(x) &= \int_2^x \frac{du}{\log u} + O(xe^{-c_0\sqrt{\log x}}),
\end{aligned}$$

Theorem 1.1 and Lemma 4.1, we know that Corollary 1.2 holds.

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References

- [1] A. Balog, *On a variant of the Piatetski-Shapiro prime number theorem*, Seminaire de Theorie Analytique des Nombres de Paris, Paris, 1986.
- [2] K. Buriev, *Additive problems with prime numbers*, Moscow: Thesis, Moscow University, 1989 (in Russian).
- [3] X. D. Cao & W. G. Zhai, *The distribution of square-free numbers of the form $[n^c]$* , Journal de theorie nombres de Bordeaux, **10** (1998), 287–299.
- [4] X. D. Cao & W. G. Zhai, *Multiple exponential sums with monomials*, Acta Arith., **92** (2000), 195–213.
- [5] X. D. Cao & W. G. Zhai, *On the Distribution of Square-Free Numbers of the Form $[n^c]$ (II)*, Acta Math. Sinica (Chin. Ser.), **51**(6) (2008), 1187–1194.
- [6] J. M. Deshouillers, *Sur la repartition des nombres $[n^c]$ dans les progressions arithmetiques*, C. R. Acad. Sci. Paris Ser. A, **277** (1973), 647–650.
- [7] S. W. Graham & G. Kolesnik, *Van der Corput's Method of Exponential Sums*, Cambridge University Press, 1991.
- [8] D. R. Heath-Brown, *The Pjateckiĭ-Šapiro prime number theorem*, J. Number Theory, **16** (1983), 242–266.
- [9] I. I. Piatetski-Shapiro, *On the distribution of prime numbers in sequences of the form $[f(m)]$* , Mat. Sb., **33** (1953), 559–566.
- [10] J. Rivat & J. Wu, *Prime numbers of the form $[n^c]$* , Glasgow Math. J., **43** (2001), 237–254.
- [11] G. J. Rieger, *Remark on a paper of Stux concerning squarefree numbers in non-linear sequences*, Pacific J. Math., **78** (1978), 241–242.

- [12] O. Robert & P. Sargos, *Three-dimensional exponential sums with monomials*, J. Reine Angew. Math., **591** (2006), 1–20.
- [13] I. E. Stux, *Distribution of squarefree integers in non-linear sequences*, Pacific J. Math., **75** (1975), 577–584.
- [14] J. D. Vaaler, *Some extremal functions in Fourier analysis*, Bull. Amer. Math. Soc., **12**(2) (1985), 183–216.